



q-TANGENT LINES OF q-REGULAR CURVES

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ABSTRACT

In this paper the notion of q-regular curves in the space is introduced and for such curves the concept q-tangent line and natural q-parametrization is investigated.

KEYWORDS: q-derivative, q-curves, natural q-parametrization

1. INTRODUCTION:

Calculus is any theory in which there is a differentiation $f \rightarrow f'$ and integration $f \rightarrow \int f$ operation in such a way that $(\int f)' = f$. This does not necessarily mean that the notion of derivative or integral is the classical notions.

A quantum calculus is a version of calculus in which we do not take limits. Derivatives are differences and anti derivatives are sums. It is a theory, where smoothness is no more required[1].

The general idea in this paper is to study q-curves in $\mathbb{R}^3$ which have velocity vector at any point that constructed by q-derivative.

2. PRELIMINARIES:

Consider an arbitrary function $f(x)$. The q-derivative $D_q f$ of the function $f(x)$ is given by

$$(D_q f)(x) = \frac{f(qx) - f(x)}{qx - x},$$

if $x \neq 0$ and $(D_q f)(0) = f'(0)$ provided $f'(0)$ exists. Note that

$$\lim_{q \rightarrow 1} D_q f(x) = \frac{d}{dx} f(x)$$

if $f(x)$ differentiable[2]. The Leibniz notation $d/dx f(x)$, a ratio of two "infinitesimals" is rather confusing, since the notion of the differential $df(x)$ requires an elaborate explanation. In contrast, the notion of q-differential is obvious and plain ratio[3].

It is clear that as with ordinary derivative, the action of taking the q-derivative of a function is a linear operator. In other words, for any constants a and b, we have

$$D_q \{af(x) + bg(x)\} = aD_q f(x) + bD_q g(x)$$

The formulas for the q-derivative of a product and a quotient of functions are [3]

$$D_q \{f(x)g(x)\} = D_q f(x)g(x) + f(qx)D_q g(x)$$

and

$$D_q \left\{ \frac{f(x)}{g(x)} \right\} = \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(qx)g(x)}.$$

If f is q-differentiable at x, then

$$f(qx) = f(x) + (q-1)x D_q f(x).$$

The q-analogue of the chain rule is more complicated since it involves q-derivatives for different values of q depending on the composed functions. The chain rule for general functions $f(x)$ and $g(x)$ is [3]

$$D_q (f \circ g)(x) = D_{\frac{g(qx)}{g(x)}} f(g(x)) D_q g(x).$$

Notice that the series on the right-hand side is guaranteed to be convergent as soon as the function f is such that for some $C > 0, \alpha > -1, |f(x)| < Cx^\alpha$ in a right neighbourhood of $x=0$.

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x.$$

Then fundamental theorem of quantum calculus is given follow[2]:

Theorem 2.1

i) If F is any anti q-derivative of the function f , namely, $D_q F = f$, continuous at $x=0$, then

$$\int_0^a f(x) d_q x = F(a) - F(0)$$

ii) For any function f one has

$$D_q \int_0^x f(t) d_q t = f(x)$$

iii) If $u(x) = \alpha x^\beta$, change of variable formula is

$$\int_{u(a)}^{u(b)} f(u) d_q u = \int_a^b f(u(x)) D_{q^{1/\beta}} u(x) d_{q^{1/\beta}} x. [2]$$

3. Q-CURVES AND Q-TANGENT LINES

Let I be an interval in the real line $\mathbb{R}$. One can picture a curve in $\mathbb{R}^3$ as a trip taken by a moving point α . At each t time in some interval, α is located at the point $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$ in $\mathbb{R}^3$. In rigorous term then α is a function from I to $\mathbb{R}^3$, and the real valued functions $\alpha_1, \alpha_2, \alpha_3$ are its Euclidean coordinate functions[4].

We define the function α to be q-differentiable provided its coordinate functions are q-differentiable.

Definition 3.1 A q-curve in $\mathbb{R}^3$ a q-differentiable functions $\alpha: I \rightarrow \mathbb{R}^3$ from a interval I into $\mathbb{R}^3$.

Definition 3.2 Let $\alpha: I \rightarrow \mathbb{R}^3$ be a q-curve in $\mathbb{R}^3$ with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. For each number t in I , the average velocity vector of α at t is the q-tangent vector

$$\alpha^q(t) = (D_q \alpha_1(t), D_q \alpha_2(t), D_q \alpha_3(t))_{\alpha(t)}$$

Definition 3.3 Let $\alpha: I \rightarrow \mathbb{R}^3$ be a q-curve in $\mathbb{R}^3$ and if $h: J \rightarrow I$ is a q-differentiable function, then the composite function $\beta: \alpha(h): J \rightarrow \mathbb{R}^3$ is a curve called a q-reparametrization of α by h .

Lemma 3.1 If β is the q-reparametrization of α by h , then

$$\beta^q(s) = \alpha^{q*}(h(s)) D_q h(s)$$

where $q^* = \frac{h(qs)}{h(s)}$.

Proof: If $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, then

$$\beta(s) = \alpha(h(s)) = (\alpha_1(h(s)), \alpha_2(h(s)), \alpha_3(h(s))).$$

Using the prime notation for derivatives, the chain rule for a composition of real

$$\text{functions } f \text{ and } g \text{ reads } D_q (f \circ g)(x) = D_{\frac{g(qx)}{g(x)}} f(g(x)) D_q g(x).$$

Thus, in the case at hand,

$$D_q (\alpha_i \circ h)(s) = D_{\frac{h(qs)}{h(s)}} \alpha_i(h(s)) D_q h(s), \quad i = 1, 2, 3.$$

By the definition of velocity, this yields

$$\beta^q(s) = \alpha^{q*}(h(s)) D_q h(s)$$

$$\text{where } q^* = \frac{h(qs)}{h(s)}.$$

Definition 3.4 A q -curve $\alpha: I \rightarrow \mathbb{R}^3$ said to be regular if $\alpha^q(t) \neq 0$ for all t in I .

From now on we shall consider only regular q -curves.

Definition 3.5 Let $\alpha: I \rightarrow \mathbb{R}^3$ be a q -curve in $\mathbb{R}^3$. $P_0 = \alpha(t_0)$ be a point in α and L be line through $P_0 = \alpha(qt_0)$. Let P be any point on α . Denote the distance of the point P from the point P_0 , and the distance of P from the line L by d and l , respectively. If $(l/d) \rightarrow 0$ as $P \rightarrow P_0$, then we say that L is the q -tangent line to the curve α at the point P_0 .

If the curve has the q -tangent line at the point P_0 , then the line PP_0^q converges to L as $P \rightarrow P_0$. Conversely, if the line PP_0^q converges to some line as $P \rightarrow P_0$, then this limit line will be the q -tangent line at P_0 [5].

Theorem 3.1 Every q -regular curve α has a q -tangent line at the point P_0 , that the vector $\alpha^q(t_0)$ its direction vector.

Proof: Suppose the curve α has the q -tangent line L at the point $P_0 = \alpha(t_0)$. Let $\vec{u}$ be a unit vector on the line L . The distance of the point $P = \alpha(t)$ from the point P_0 is

$$d = \|\overrightarrow{\alpha(t)\alpha(qt_0)}\|. \text{ Furthermore, the distance point } P \text{ from the line } L \text{ is}$$

$l = \|\overrightarrow{\alpha(t)\alpha(qt_0)} \wedge \vec{u}\|$, where $\wedge$ denotes the vector product. By the definition of the q -tangent line, we have

$$\frac{l}{d} = \frac{\|\overrightarrow{\alpha(t)\alpha(qt_0)} \wedge \vec{u}\|}{\|\overrightarrow{\alpha(t)\alpha(qt_0)}\|} \rightarrow 0, \quad (t \rightarrow t_0).$$

On the other hand, as $t \rightarrow t_0$

$$\frac{l}{d} = \left\| \frac{\overrightarrow{\alpha(qt_0) - \alpha(t)}}{qt_0 - t} \wedge \vec{u} \right\| \rightarrow \frac{\|\overrightarrow{\alpha^q(t_0)} \wedge \vec{u}\|}{\|\overrightarrow{\alpha^q(t_0)}\|} = 0.$$

Therefore $\overrightarrow{\alpha^q(t_0)} \wedge \vec{u} = 0$. Since $\overrightarrow{\alpha^q(t_0)} \neq 0$, it follows that the vectors $\overrightarrow{\alpha^q(t_0)}$

and $\vec{u}$ are collinear. Thus if the q -tangent line exists, then it has $\overrightarrow{\alpha^q(t_0)}$ as a direction vector and it is unique.

Conversely, let L be a line through the point P_0 and has the vector $\overrightarrow{\alpha^q(t_0)}$ as a direction vector. Then the line L will be q -tangent line at the point P_0 . Indeed as above, we have

$$\frac{l}{d} = \left\| \frac{\overrightarrow{\alpha(t)\alpha(qt_0)} \wedge \frac{\overrightarrow{\alpha^q(t_0)}}{\|\overrightarrow{\alpha^q(t_0)}\|}}{\overrightarrow{\alpha(t)\alpha(qt_0)}} \right\| \rightarrow \frac{\|\overrightarrow{\alpha^q(t_0)} \wedge \frac{\overrightarrow{\alpha^q(t_0)}}{\|\overrightarrow{\alpha^q(t_0)}\|}\|}{\|\overrightarrow{\alpha^q(t_0)}\|^2} = 0.$$

Then from the Theorem 3.1, it follows that the equation of the q -tangent line at the point $P_0 = \alpha(t_0)$ will be

$$\frac{x - \alpha_1(t_0)}{\alpha_1^q(t_0)} = \frac{y - \alpha_2(t_0)}{\alpha_2^q(t_0)} = \frac{z - \alpha_3(t_0)}{\alpha_3^q(t_0)}.$$

4. NATURAL Q-PARAMETRIZATIONS OF Q-CURVES:

Let $\alpha: I \rightarrow \mathbb{R}^3$ be a q -curve in $\mathbb{R}^3$ with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Introduce the q -arc length function by the formula

$$s(t) = \int_a^t \|\alpha^q(\tau)\| d_q \tau$$

where

$$\|\alpha^q(\tau)\| = \sqrt{\sum_{i=1}^3 D_q \alpha_i(\tau)}.$$

The function $s(t)$ is strictly increasing and continuous. The variable s can be used as a parameter for the curve. Such a parametrization of a q -curve we call natural q -parametrization.

Then the variable t becomes a function of s , $t=t(s)$. Therefore we have $\alpha(t) = \alpha(t(s))$.

Theorem 4.1 Let $\alpha: I \rightarrow \mathbb{R}^3$ be a q -curve. In the case of natural q -parametrization of the curve, the q -tangent vector $\alpha^q(t)$ is a unit vector.

Proof: By the definition of the q -arc length function, we have

$$D_q s(t) = \|\alpha^q(t)\| > 0, \quad t \in I.$$

Then from the chain rule we get

$$\alpha^q(t) = \alpha^{q*}(s) D_q s(t) = \alpha^{q*}(s) \|\alpha^q(t)\|$$

and hence

$$\alpha^{q*}(s) = \frac{\alpha^q(t)}{\|\alpha^q(t)\|}.$$

Consequently $\|\alpha^{q*}(s)\| = 1$.

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